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Anomalous diffusion on the Hanoi networks

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Abstract – Diffusion is modeled on the recently proposed Hanoi networks by studying the mean-square displacement of random walks with time, $\langle r^2\rangle \sim t^{2/d_w}$. It is found that diffusion —the quintessential mode of transport throughout Nature— proceeds faster than ordinary, in one case with an exact, anomalous exponent $d_w = 2 - \log_2(\phi) = 1.30576\ldots$. It is an instance of a physical exponent containing the "golden ratio" $\phi = (1+\sqrt{5})/2$ that is intimately related to Fibonacci sequences and since Euclid's time has been found to be fundamental throughout geometry, architecture, art, and Nature itself. It originates from a singular renormalization group fixed point with a subtle boundary layer, for whose resolution ϕ is the main protagonist. The origin of this rare singularity is easily understood in terms of the physics of the process. Yet, the connection between network geometry and the emergence of ϕ in this context remains elusive. These results provide an accurate test of recently proposed universal scaling forms for first passage times.

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Introduction. – The study of anomalous diffusion is an integral part in the analysis of transport processes in complex materials [1–5]. Random environments often slow transport significantly, leading to sub-diffusive behavior. Much attention has thus been paid to model sub-diffusion on designed structures with some of the trappings of disordered materials, exemplified by refs. [6–11]. Even self-organized critical processes can be shown to evolve sub-diffusively, controlled by the memory of all past events [12]. On the other hand, tracer particles in rapidly driven fluids may exhibit super-diffusive behavior [13], typically modeled in terms of Lévy flights [3,14]. Both regimes are self-similar, fractal generalizations of ordinary diffusion.

In this letter we consider diffusion on two new networks, which yield interesting realizations of super-diffusive behavior. Both of these networks were introduced to explore certain aspects of small-world behavior [15]. Their key distinguishing characteristic is their ability to mix a geometric backbone, *i.e.* a one-dimensional lattice, with small-world links in a non-random, hierarchical structure. In particular, these networks permit a smooth interpolation between finite-dimensional and mean-field properties, which is absent from the renormalization group (RG) due to Migdal and Kadanoff, for instance [16]. The unusual structure of these networks recasts the RG into a novel form, where the equations are essentially those of a one-dimensional model in which the complex hierarchy enters at each RG-step as a (previously unrenormalized) source term. This effect is most apparent in the real-space RG for the Ising models discussed in ref. [15]. It is obscured in our dynamic RG treatment below, since these walks are always embedded on the lattice backbone. On the practical side, their regular, hierarchical structure allows for easily engineered implementations, say, to efficiently synchronize communication networks [15]. Regarding diffusion, one of the networks proves to be merely an incarnation of a Weierstrass random walk found for Lévy flights [14] with ballistic transport, while the other network shows highly non-trivial transport properties, very much unlike a Lévy flight, as revealed by our exact RG treatment. The fixed point equations are singular and exhibit a boundary layer [17]. It provides a tangible case of a singularity in the RG [10,18] that is easily interpreted in terms of the physics.

Generating Hanoi networks. – In the Tower-of-Hanoi problem [19], disks of increasing size, labeled $i = 1$ to k from top to bottom, are stacked up and have to be moved in a Sisyphean task into a 2nd stack, disk-by-disk, while at no time a larger disk can be placed onto a smaller

[(a)] URL: http://www.physics.emory.edu/faculty/boettcher/.

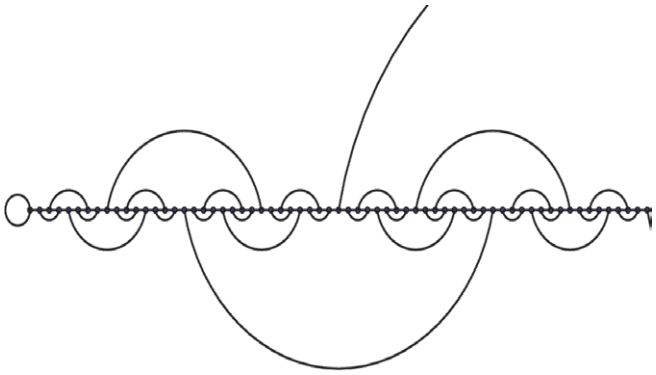


Fig. 1: Depiction of the planar "Tower-of-Hanoi" network HN3. Here, the $1d$-backbone of sites extends over $0 \leqslant n < \infty$.

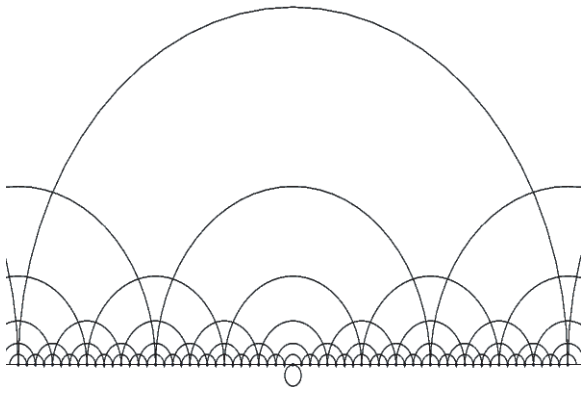


Fig. 2: Depiction of the "Tower-of-Hanoi" network HN4. Here, the $1d$-backbone of sites extends over $-\infty < n < \infty$. The site $n = 0$, not covered by eq. (1), is special and is connected to itself here.

one. To this end, a 3rd stack is provided as overflow. First, disk 1 moves to the overflow and disk 2 onto the 2nd stack, followed by disk 1 on top of 2. Now, disk 3 can move to the overflow, disk 1 back onto disk 4, disk 2 onto 3, and 1 onto 2. Now we have a new stack of disks 1, 2, and 3 in prefect order, and only $k-3$ more disk to go! But note the values of disk-label i in the sequence of moves: 1-2-1-3-1-2-1-4-1-2-1-3-1-2-1-5-..., and so on.

Inspired by models of ultra-slow diffusion [6,7], we create our networks as follows. First, we lay out this sequence on a $1d$-line of nearest-neighbor connected sites labeled from $n = 1$ to $n = L = 2^k - 1$ (the number of moves required to finish the problem). In general, any site $n(\neq 0)$ can be described uniquely by

$$n = 2^{i-1}(2j+1), \tag{1}$$

where i is the label of the disk moved at step n in the sequence above and $j = 0, 1, 2 \ldots$. To wit, let us further connect each site n to the closest site n' that is 2^i steps away and possesses the same value of i, both only having a site of value at most $i+1$ between them. According to the sequence, site $n = 1$ (with $i = 1$) is now also connected to $n' = 3$, 5 to 7, 9 to 11, etc. For sites with $i = 2$, site $n = 2$ now also connects to $n' = 6$, 10 to 14, 18 to 22, etc, and so on also for $i > 2$. As a result, we get the network depicted in fig. 1 that we call HN3. Except at the boundary, each site now has three neighbors, left and right along the $1d$ "backbone" and a 3rd link to a site 2^i steps away. If we further connect each site also to a fourth site 2^i steps in the other direction and allow $j = 0, \pm 1, \pm 2, \ldots$, we obtain the network in fig. 2, called HN4, where each site now has four neighbors.

These new "Tower-of-Hanoi" networks —a mix of local, geometric connections and "small-world"-like long-range jumps— has fascinating properties. It is recursively defined with obvious fractal features. A collection of the structural and dynamic features of HN3 and HN4 are discussed in ref. [15], such as results for Ising models and synchronization.

Diffusion on the Hanoi networks. – To model diffusion on these networks, we study simple random walks with nearest-neighbor jumps along the available links, but using the one-dimensional lattice backbone as our metric to measure distances, which implies a fractal dimension of $d_f = 1$. Embedded in that space, we want to calculate the non-trivial diffusion exponent d_w defined by the asymptotic mean-square displacement

$$\langle r^2 \rangle \sim t^{2/d_w}. \tag{2}$$

A more extensive treatment yielding also first-return probabilities is given elsewhere [20].

First, we consider a random walk on HN4. The "master-equation" [21] for the probability of the walker to be at site n, as defined in eq. (1), at time t is given by

$$\begin{aligned} \mathcal{P}_{n,t} \;=\; & \frac{1-p}{2}\left[\mathcal{P}_{n-1,t-1} + \mathcal{P}_{n+1,t-1}\right] \\ & + \frac{p}{2}\left[\mathcal{P}_{n-2^i,t-1} + \mathcal{P}_{n+2^i,t-1}\right], \end{aligned} \tag{3}$$

where p is the probability to make a long-range jump. (Throughout this letter, we considered p uniform, independent of n or t.) A detailed treatment of this equation in terms of generating functions is quite involved and proved fruitless, as will be shown elsewhere [20]. Instead, we note that the long-time behavior is dominated by the long-range jumps, as discussed below for HN3. To simplify matters, we set $p = 1/2$ here, although any other finite probability should lead to the same result. We make an "annealed" approximation, *i.e.*, we assume that we happen to be at some site n in eq. (1) with probability $1/2^i$, corresponding to the relative frequency of such a site, yet independent of update-time or history. This ignores the fact that in the network geometry a long jump of length 2^i can be followed *only* by another jump of that length or a jump of unit length, and that many intervening steps are necessary to make a jump of length 2^{i+1}, for instance.

Here, at each instant the walker jumps a distance 2^i left or right irrespectively with probability $1/2^i$, and we can write

$$\mathcal{P}_{n,t} = \sum_{n'} T_{n,n'} \mathcal{P}_{n',t-1} \tag{4}$$

with

$$T_{n,n'} = \frac{a-1}{2a} \sum_{i=0}^{\infty} a^{-i} \left(\delta_{n-n',b^i} + \delta_{n-n',-b^i} \right), \tag{5}$$

where $a = b = 2$. Equations (4), (5) are identical to the Weierstrass random walk discussed in refs. [14,22] for arbitrary $1 < a < b^2$. There, it was shown that $d_w = \ln(a)/\ln(b)$, which leads to the conclusion that $d_w = 1$ in eq. (2) for HN4, as has been predicted (with logarithmic corrections) on the basis of numerical simulations in ref. [15]. These logarithmic corrections are typical for walks with marginal recurrence, which typically occurs when $d_w = d_f$, such as for ordinary diffusion in two dimensions [4].

For HN3, the master-equation in the bulk reads for

$$\begin{aligned} \mathcal{P}_{n,t} &= \frac{1-p}{2} \left[\mathcal{P}_{n-1,t-1} + \mathcal{P}_{n+1,t-1} \right] + p\, \mathcal{P}_{n',t-1}, \\ n' &= \begin{cases} n + 2^i, & j \text{ even}, \\ n - 2^i, & j \text{ odd}, \end{cases} \end{aligned} \tag{6}$$

with n as in eq. (1), and p as before.

In the RG [21,23] solution of eq. (6), at each step we eliminate all odd sites, *i.e.*, those sites with $i = 1$ in eq. (1). As shown in fig. 3, the elementary unit of sites effected is centered at all sites n having $i = 2$ in eq. (1). We know that such a site n is surrounded by two sites of odd index, which are mutually linked. Furthermore, n is linked by a long-distance jump to a site also of type $i = 2$ at $n \pm 4$ in the neighboring elementary unit, where the direction does not matter here. The sites $n \pm 2$, which are shared at the boundary between such neighboring units also have even index, but their value of $i \geqslant 3$ is indetermined and irrelevant for the immediate RG step, as they have a long-distance jump to some sites $m_\pm$ at least eight sites away.

Using a standard generating function [21],

$$x_n(z) = \sum_{t=0}^{\infty} \mathcal{P}_{n,t}\, z^t, \tag{7}$$

yields for the five sites inside the elementary unit centered at n:

$$\begin{aligned} x_n &= a\,(x_{n-1} + x_{n+1}) + c\,(x_{n-2} + x_{n+2}) + p_2\, x_{n\pm4}, \\ x_{n\pm1} &= b\,(x_n + x_{n\pm2}) + p_1\, x_{n\mp1}, \\ x_{n\pm2} &= a\,(x_{n\pm1} + x_{n\pm3}) + c\,(x_n + x_{n\pm4}) + p_2\, x_{m_\pm}, \end{aligned} \tag{8}$$

where we have absorbed the parameters p and z into general "hoping rates" that are initially $a^{(0)} = b^{(0)} = \frac{z}{2}(1-p)$, $c^{(0)} = 0$, and $p_1^{(0)} = p_2^{(0)} = zp$.

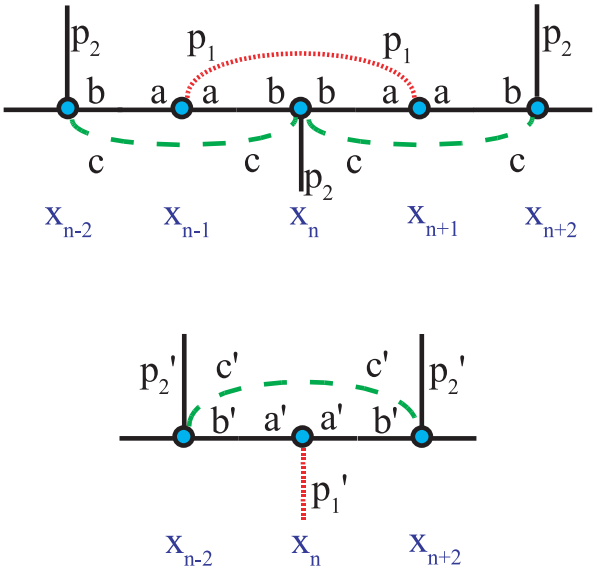


Fig. 3: Depiction of the (exact) RG step for random walks on HN3. Hopping rates from one site to another along a link are labeled at the originating site. The RG step consists of tracing out odd-labeled variables $x_{n\pm1}$ in the top graph and expressing the renormalized rates (a', b', c', p_1', p_2') on the right in terms of the previous ones (a, b, c, p_1, p_2) on the bottom. The node x_n, bridged by a (dotted) link between x_{n-1} and x_{n+1}, is special as it *must* have $n = 2(2j+1)$ and is to be decimated at the following RG step, justifying the designation of p_1' and a'. Note that the original graph in fig. 1 does not have the green, dashed links with hopping rates (c, c'), which *emerge* during the RG recursion.

The RG update step consist of eliminating from these five equations those two that refer to an odd index, $n \pm 1$. After some algebra, we obtain

$$\begin{aligned} x_n &= b'(x_{n-2} + x_{n+2}) + p_1'\, x_{n\pm4}, \\ x_{n\pm2} &= a'(x_n + x_{n\pm4}) + c'(x_{n\mp2} + x_{n\pm6}) + p_2'\, x_{m_\pm}, \end{aligned} \tag{9}$$

with

$$\begin{aligned} a' &= \frac{[ab + c(1-p_1)](1+p_1)}{1 - p_1^2 - 2ab}, \\ b' &= \frac{ab + c(1-p_1)}{1 - p_1 - 2ab}, \\ c' &= \frac{abp_1}{1 - p_1^2 - 2ab}, \\ p_1' &= \frac{p_2(1-p_1)}{1 - p_1 - 2ab}, \\ p_2' &= \frac{p_2(1-p_1^2)}{1 - p_1^2 - 2ab}. \end{aligned} \tag{10}$$

If for all sites $l = n,\ n \pm 2, n \pm 4, \ldots$ in eq. (9) we further identify[1] $x_l = C\, x'_{l/2}$, we note that the primed equations

[1] As we will show elsewhere [20], the constant C is determined when initial and boundary conditions are considered, as is essential for the case of first transit and return times [21].

coincide with the unprimed ones in eqs. (8). Hence, the RG recursion equations in (10) are *exact* at any step k of the RG, where unprimed quantities refer to the k-th recursion and primed ones to $k+1$.

Solving eqs. (10) algebraically at infinite time (which corresponds to the limit $z \nearrow 1$, see eq. (7)) and for $k+1 \sim k \to \infty$ (by dropping the prime on all left-hand parameters), we —apparently— obtain only two fixed points at $a=b=1/2$ and $c=p_1=p_2=0$, and $a=b=c=0$ and $p_1=p_2=1$. The first fixed point corresponds to an ordinary $1d$ walk without long-range jumps, in the second there is no hopping along the $1d$-backbone at all and the walker stays *confined,* jumping back-and-forth within a single, long-range jump. Yet, both fixed points are *unstable* with respect to small perturbations in the initial parameters.

Starting with any positive probability p for long-range jumps, those dominate over the $1d$ walk at long times. Paradoxically, exclusive long-range jumps found at the 2nd fixed point lead to confinement, itself undermined by *any* positive probability to escape along the $1d$-line, allowing to reach even longer jumps. Instead, the process gets attracted to a third, stable fixed point hidden inside a singular *boundary layer* [17] in the renormalization group equations (10) near the confined state.

We have to account for the asymptotic boundary layer in eqs. (10) with the Ansatz $y \sim A_y \alpha^{-k} \to 0$ for $y \in \{a, b, c, 1-p_1, 1-p_2\}$, where $k \to \infty$ refers to the k-th RG step. Choosing $A_a = 1$, the other A_y's and the eigenvalues α are determined *self-consistently.* The only eigenvalue satisfying the requirement $\alpha > 1$ is $\alpha = 2/\phi$. Here, $\phi = \left(\sqrt{5}+1\right)/2 = 1.6180\ldots$ is the legendary "golden ratio" [24] defined by Euclid [25]. Hence, every renormalization of network size, $L \to L' = 2L$, has to be matched by a rescaling of hopping rates with $\alpha = 2/\phi$ to keep motion along the $1d$-backbone finite and prevent confinement.

Extending the analysis to include finite-time corrections (*i.e.*, $1 - z \ll 1$), we generalize the above Ansatz to

$$y^{(k)} \sim A_y \alpha^{-k} \left\{1 + (1-z)\, B_y \beta^k + \ldots\right\} \tag{11}$$

for all $y \in \{a, b, c, 1-p_1, 1-p_2\}$. In addition to the leading-order constants A_y and α, also the next-leading constants are determined self-consistently, and we extract uniquely $\beta = 2\alpha$. Accordingly, time re-scales now as

$$T \to T' = 2\alpha T, \tag{12}$$

and we obtain from eq. (2) with $T \sim L^{d_w}$ for the diffusion exponent for HN3

$$d_w = 2 - \log_2 \phi = 1.30576\ldots. \tag{13}$$

The result for d_w is in excellent agreement with our simulations, as shown in fig. 4.

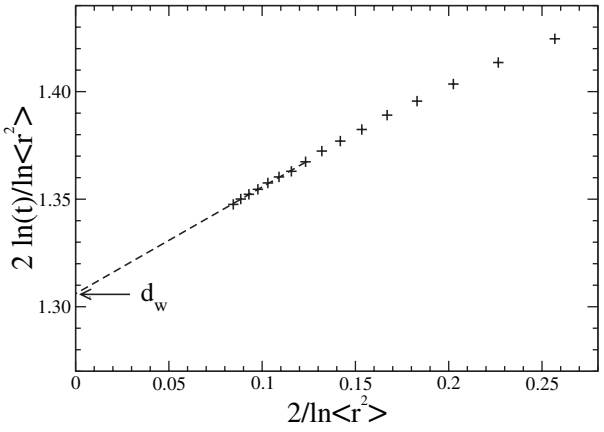


Fig. 4: Plot of the results from simulations of the mean-square displacement of random walks on HN3 displayed in fig. 1. More than 10^7 walks were evolved up to $t_{\max} = 10^6$ steps to measure $\langle r^2 \rangle_t$. The data is extrapolated according to eq. (2), such that the intercept on the vertical axis determines d_w asymptotically. The exact result from eq. (13) is indicated by the arrow.

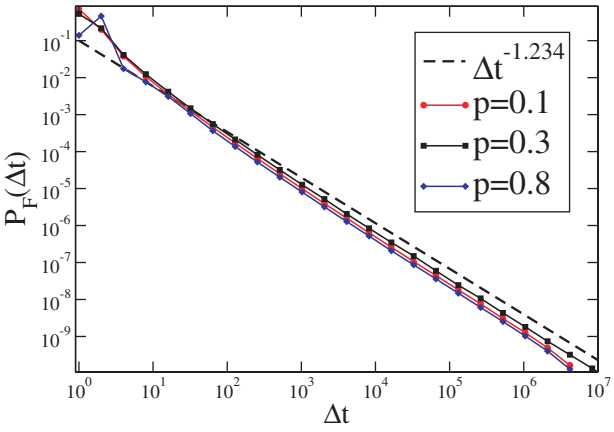


Fig. 5: Plot of the probability $P_F(\Delta t)$ of first returns to the origin after Δt update steps on a system of unlimited size. Data was collected for three different walks on HN3 with $p = 0.1$ (circles), $p = 0.3$ (squares), and $p = 0.8$ (diamonds). The data with the smallest and largest p exhibit strong short-time effects. The exact result in eq. (14), $\mu = 1.234\ldots$, is indicated by the dashed line.

Using the methods from ref. [21], a far more extensive treatment shows [20] that the exponent μ for the probability distribution, $P_F(\Delta t) \sim \Delta t^{-\mu}$, of first-return times Δt is given by

$$\mu = 2 - \frac{1}{d_w} = 1.2342\ldots. \tag{14}$$

The relation between μ and d_w is typical also for Lévy flights [3], and the result is again borne out by our simulations, see fig. 5. It is remarkable, though, that the more detailed analysis in ref. [20] also shows that walks on HN3 are *not* uniformly recurrent, as the result of

$d_w > d_f = 1$ here would indicate. That calculation shows that only sites on the highest level of the hierarchy are recurrent. While all other sites do share the same exponent μ in eq. (14) for actual recurrences, they have a diminishing return probability with decreasing levels in the infinite system limit. This is clearly a consequence of walkers being nearly confined to the highest levels of the hierarchy at long times, as expressed by the boundary layer.

We finally contrast the behavior of HN4 discovered above with the analysis of HN3. Clearly, when long-range jumps are interconnected as in HN4, there is no confinement, the boundary layer disappears (which would be similar to $\alpha = 1$ in eqs. (11), (12) for HN3), and diffusion spreads ballistically, $d_w = 1$. Our numerical studies, and the similarity to Weierstrass random walks [22], further supports that μ for walks on HN4 is also given by eq. (14), leading to $\mu = 1$. This scaling is again indicative of a marginally recurrent state and requires logarithmic corrections for proper normalization, as was observed in simulations [15].

Conclusions. – We conclude with two further considerations. First, in reference to the potential of these networks to interpolate between long-range and a finite-dimensional behavior that we invoked in the introduction, we just add the following illustrative remark: If the probability to undertake a long-distance jump would be distance-dependent in each level of the hierarchy, we can obtain immediately a new result for walks on HN4 in the annealed approximation above. Let p vary with a power of the backbone-distance between sites, say $p \propto r^{-\sigma}$, then for each level i of the hierarchy it is $r = r_i = 2^i$, *i.e.* $p = p_i \propto 2^{-i\sigma}$, and the weight to make a jump of length 2^i in eq. (5) is given by $a = 2^{1+\sigma}$, leading to $d_w = 1 + \sigma$. As can be expected, the analysis of the Weierstrass walk breaks down for $\sigma \to 1^-$, at which point the long-range jumps become irrelevant and we obtain the results for ordinary $1d$ diffusion. Hence, $0 \leqslant \sigma \leqslant 1$ interpolates analytically between long-range and one-dimensional behavior of the random walk on HN4. (In fact, the analysis formally can be extended to $0 > \sigma > -1$, where the walk becomes non-recurrent and is dominated by high levels in the hierarchy. Yet, the annealed approximation that assumes free transitions between different levels of the hierarchy is bound to fail.)

Our final consideration concerns a recent proposal by Condamin *et al.* [5] for a very general scaling form for mean first-passage times $\langle T \rangle$ for walks as a function of distance r between starting and target site on a graph (lattice, network, etc.) of N sites. Based on $d_w - d_f$, ref. [5] determined that

$$\langle T \rangle \sim N \begin{cases} A + B r^{d_w - d_f}, & d_w > d_f, \\ A + B \ln(r), & d_w = d_f, \\ A - B r^{d_w - d_f}, & d_w < d_f, \end{cases} \qquad (15)$$

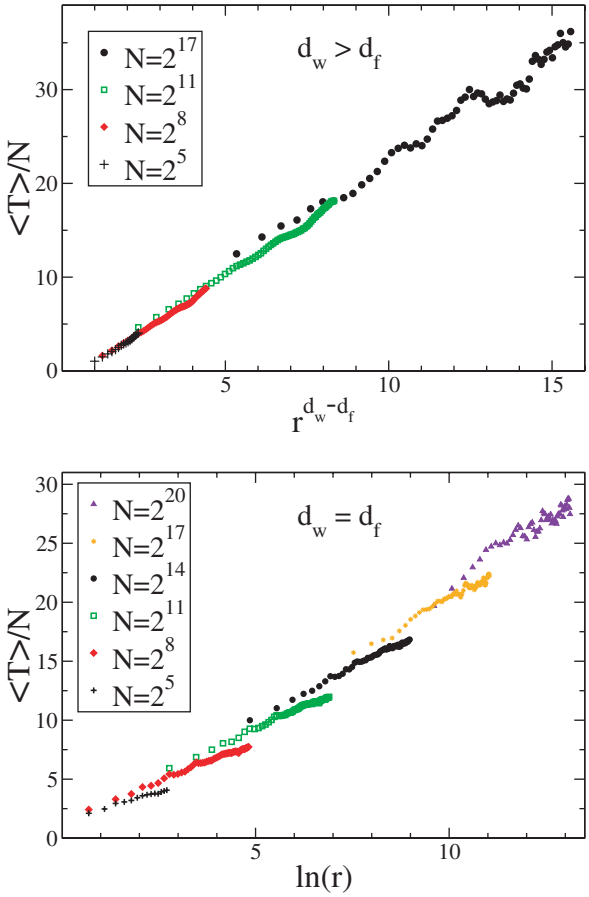


Fig. 6: Plot of the mean first-passage times $\langle T \rangle$ for walks as a function of distance r between starting and target site for HN3 (top) and HN4 (bottom). The data has been scaled according to eq. (15) such that the data collapses asymptotically onto a line that only depends on the model but that is independent of system size N. This collapse is excellent for HN3, it is somewhat weaker for HN4. Although all system sizes lead to linear forms in $\ln(r)$, their slope apparently varies with N. This could be caused by logarithmic scaling corrections to the slope, or by the lack of asymptotic behavior at the available system sizes N.

for *fixed* constants A, B, independent of N and r. Our networks provide a non-trivial set of exponents to explore these relations with simple simulations. In particular, HN3 with $d_w - d_f = 0.30576$ provides an instance for a power-law–divergent mean first-passage time, while HN4 exactly probes the marginal case $d_w = d_f (= 1)$ with a logarithmic divergence of $\langle T \rangle$. When plotting $\langle T \rangle / N$ as a function of $r^{d_w - d_f}$ or $\ln(r)$, respectively, in figs. 6 we indeed obtain a universal straight line over many orders of magnitude in N and r, indicative of fixed A, B.

* * *

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