The peculiar phase structure of random graph bisection

Allon G. Percus,[1,a)] Gabriel Istrate,[2,b)] Bruno Gonçalves,[3,c)]
Robert Z. Sumi,[4,d)] and Stefan Boettcher[3,e)]
[1]*Information Sciences Group, Computer, Computational and Statistical Sciences Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA and Department of Mathematics, UCLA, Los Angeles, California 90095, USA*
[2]*eAustria Research Institute, Bd. V. Pârvan 4, cam. 045B, RO-300223 Timişoara, Romania*
[3]*Department of Physics, Emory University, Atlanta, Georgia 30322-2430, USA*
[4]*Department of Physics, Babes-Bolyai University, RO-400884 Cluj, Romania and eAustria Research Institute, Bd. V. Pârvan 4, cam. 045B, RO-300223 Timişoara, Romania*

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The mincut graph bisection problem involves partitioning the n vertices of a graph into disjoint subsets, each containing exactly $n/2$ vertices, while minimizing the number of "cut" edges with an endpoint in each subset. When considered over sparse random graphs, the phase structure of the graph bisection problem displays not only certain familiar properties but also some surprises. It is known that when the mean degree is below the critical value of 2 log 2, the cutsize is zero with high probability. We study how the minimum cutsize increases with mean degree above this critical threshold, finding a new analytical upper bound that improves considerably upon previous bounds. Combined with recent results on expander graphs, our bound suggests the unusual scenario that random graph bisection is replica symmetric up to and beyond the critical threshold, with a replica symmetry breaking transition possibly taking place *above* the threshold. An intriguing algorithmic consequence is that although the problem is NP-hard, we can conceivably find near-optimal cutsizes (whose ratio to the optimal value approaches 1 asymptotically) in polynomial time for typical instances near the phase transition. © *2008 American Institute of Physics.* [DOI: 10.1063/1.3043666]

I. INTRODUCTION

The graph bisection problem arises in a wide range of contexts, ranging from computer-aided design[1] to image processing[9,23] to protecting networks from attack.[18] The basic mincut graph bisection (or graph bipartitioning) problem is defined as follows. Given an undirected, unweighted graph, partition its n nodes into two disjoint subsets of equal size, while minimizing the number w of "cut" edges with an endpoint in each subset. This is an NP-hard optimization problem.[14]

From a statistical physics perspective, graph bisection is also equivalent to finding the ground state of a ferromagnet constrained to have zero magnetization.[13,22,27,24,25] Over the past decade, the study of combinatorial optimization problems within the physics community as well as the study of statistical physics approaches within the computer science community has led to remarkable successes.[32] By considering optimization problems over appropriately parametrized ensembles of random instances, physicists have developed methods leading both to analytical predictions of global optima and to algorithmic approaches for finding these. One particularly rich source of insight has been the investigation of phase transition behavior: for many combinatorial optimiza-

[a)]Electronic mail: percus@ipam.ucla.edu.
[b)]Electronic mail: gabrielistrate@acm.org.
[c)]Electronic mail: bgoncalves@physics.emory.edu.
[d)]Electronic mail: rsumi@phys.ubbcluj.ro.
[e)]Electronic mail: stb@physics.emory.edu.

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tion problems, both the nature of the solution space and the average-case complexity of algorithms solving the problem are closely related to an underlying phase structure.

In this paper, we consider graph bisection over Erdős–Rényi (Bernoulli) random graphs[12] generated from the $\mathcal{G}_{np}$ ensemble: given n nodes, place an edge between each of the $\binom{n}{2}$ pairs of nodes independently with probability p. We are interested in the regime of sparse random graphs, where p scales as $1/n$ and so a vertex's mean degree $p(n-1)$ is some finite α. On such graphs, it is known that a phase transition occurs at $\alpha_c = 2 \log 2$.[16,27,24–26] For $\alpha < \alpha_c$, typical graphs can be bisected without cutting any edges: the *bisection width* w is zero. For $\alpha > \alpha_c$, the graph typically contains a connected component (giant component) of size greater than $n/2$ and so a bisection requires cutting edges, giving $w > 0$. In other combinatorial optimization problems such as graph coloring or satisfiability, a similar sharp threshold coincides with a rapid escalation of average-case computational complexity.[11,30,32]

Additionally, for many of these other problems, at some $\alpha_d \leq \alpha_c$ the solution space undergoes a structural transition that corresponds to replica symmetry breaking (RSB).[4,29] Define two optimal solutions to be *adjacent* if the Hamming distance between them is small: an infinitesimal fraction of the variables differs in value. Then below α_d, any two solutions are connected by a path of adjacent solutions. The solution space may be thought of as a single "cluster." This is a replica symmetric phase. But above α_d, the cluster fragments into multiple nonadjacent clusters, differing by an extensive number of variables. This is an RSB phase. In cases where $\alpha_d < \alpha_c$, such as 3-COL and 3-SAT, this structural transition may contribute to the rise in algorithmic complexity as the main threshold α_c is approached from below.[28]

Our results indicate that the scenario is quite different for graph bisection. Rather than the RSB transition taking place below α_c, our arguments suggest that it actually occurs somewhere above α_c. This would make graph bisection the first combinatorial optimization problem we are aware of that displays this behavior. Furthermore, in 1987, Liao[24,25] proposed an analytical solution for bisection width that assumes replica symmetry above α_c, so our results help validate this prediction. Finally, the same arguments that indicate replica symmetry also suggest an algorithm for finding extremely close approximations to the optimal solution, for $\alpha \in (\alpha_c, \alpha_d)$, in polynomial time.

The rest of the paper is organized as follows. In Sec. II we study the bisection width (or "cutsize") w as a function of mean degree α. We present numerical results and derive a new upper bound by analyzing a simple greedy heuristic. In Sec. III we study the solution cluster structure and argue that the problem is replica symmetric through and beyond the critical threshold $\alpha_c = 2 \log 2$. In Sec. IV we conclude with a discussion of the physical and algorithmic consequences of this unusual phase structure.

II. BISECTION WIDTH

Formally, the problem is as follows. Given an undirected, unweighted graph $G=(V,E)$, where $|V|=n$ is even, a valid solution is a partition of the vertex set into two disjoint subsets V_1 and V_2 such that $|V_1|=|V_2|=n/2$. An optimal solution is one that minimizes the bisection width $w = |E(V_1,V_2)|$, i.e., the number of cut edges with an endpoint in each subset. We consider graphs G drawn uniformly at random from $\mathcal{G}_{np}$ and take our control parameter to be the graph's mean degree $\alpha = p(n-1)$.

The $\mathcal{G}_{np}$ ensemble[12,8] has been studied very extensively since its introduction by Erdős and Rényi in 1959, and many of its structural properties are known. These have a crucial effect on the bisection width.

- For $\alpha < 1$, the graph consists only of small components, the largest being asymptotically of size $O(\log n)$. The fraction of monomers, or isolated vertices, is asymptotically almost surely (a.a.s.) itself at least e^{-1}. Consequently, with high probability a perfect bisection ($w=0$) exists.
- For $\alpha > 1$, there is a unique giant component containing $O(n)$ vertices. The second-largest component is asymptotically of size $O(\log n)$. As long as α is below the threshold α_c

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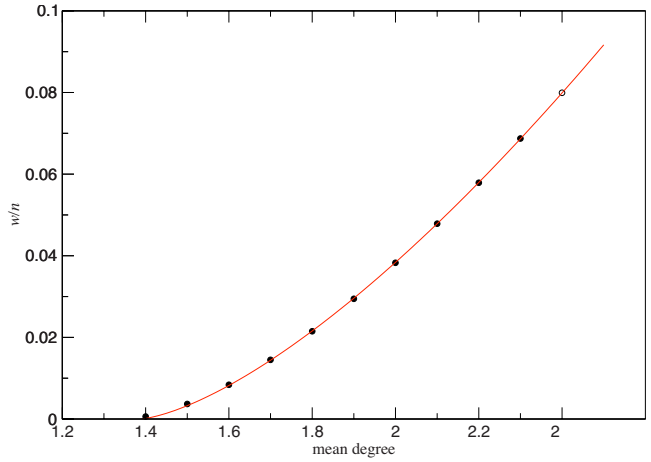


FIG. 1. (Color online) Presumed optimal bisection width, found by extremal optimization on random graphs for $n=2000$. Line shows fit to $\gamma(\alpha-\alpha_c)^{\beta}$, with $\gamma\approx 0.0783$ and $\beta\approx 1.46$. See also Ref. 5.

$=2\log 2$, the giant component a.a.s. contains a fraction $g<1/2$ of the vertices, and the fraction of monomers is still above 1/4. This allows one to prove[16,26] that with high probability, $w=0$ for all $\alpha<2\log 2$.

- For $\alpha>2\log 2$, $g>1/2$, and so bisecting the graph requires cutting some of its edges. It is known[26] that w is extensive (scales as n) in this regime, as long as α is finite.

How does the bisection width constant w/n behave for $\alpha>\alpha_c$? Figure 1 shows simulation results using the extremal optimization[6] heuristic to find the presumed optimal bisection. These suggest that the bisection width undergoes a continuous transition at α_c, and that its first derivative is continuous as well. However, neither of these properties has yet been shown analytically. We will do so now, by deriving a straightforward upper bound on w. We give the main elements of the derivation here; a more rigorous mathematical treatment will be provided elsewhere.[20]

We obtain the upper bound by "stripping" trees off of the giant component. For random graphs with mean degree α, it is known that the fraction g of nodes that are in the giant component is given a.a.s. by[12]

$$g = 1 - e^{-\alpha g}, \tag{1}$$

and the fraction b of nodes that are in trees within the giant component (the *mantle*[33]) is a.a.s.

$$b = \alpha g(1-g). \tag{2}$$

Now imagine a greedy process (Fig. 2) of removing trees, starting from the largest one and

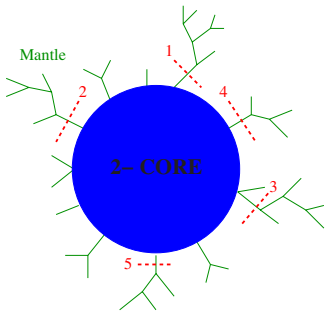


FIG. 2. (Color online) Schematic of giant component consisting of a *mantle* of trees and a remaining 2-core. Greedy upper bound algorithm cuts trees of decreasing size. Numbers show order of cuts, depicted by dashed lines.

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decreasing in size, until the giant component has been pruned to size $\leq n/2$. In order to know how many trees need to be removed, consider the distribution of tree sizes, $P(t)$. A fortunate consequence of the probabilistic independence in $\mathcal{G}_{np}$ is that $P(t)$ is simply given by the number of ways of constructing a tree of size t from q roots and r other nodes, where $q=(g-b+\sigma(n))n$ is the number of nodes in the *2-core* of the giant component (all nodes *not* in the mantle) and $r=(b+\tau(n))n$ is the number of nodes in the mantle, with $\sigma(n)$ and $\tau(n)$ being random functions that go a.a.s. to zero.[21,2] A basic combinatorial argument then gives

$$P(t) = \binom{r}{t} t^t \frac{q}{r} \frac{(q+r-t)^{r-t+1}}{(q+r)^{r-1}}, \tag{3}$$

and for large n, letting $\rho=b/g$,

$$P(t) \approx \frac{(\rho t)^t e^{-\rho t}}{t!} \frac{1-\rho}{\rho} \tag{4}$$

for $t \geq 1$.

Let z be the total number of trees. Then, given $P(t)$, the total number of nodes in all trees is $\Sigma_{t=1}^{\infty} tP(t)z$. As this is simply the size of the mantle, it holds a.a.s. that

$$\frac{z}{n} = \frac{b}{\sum_{t=1}^{\infty} tP(t)}. \tag{5}$$

Let us evaluate this expression. From normalization of $P(t)$,

$$\sum_{t=1}^{\infty} \frac{(\rho t)^t e^{-\rho t}}{t!} = \frac{\rho}{1-\rho}, \tag{6}$$

and differentiating with respect to ρ,

$$\left(\frac{1}{\rho} - 1\right) \sum_{t=1}^{\infty} \frac{(\rho t)^t e^{-\rho t}}{(t-1)!} = \frac{1}{(1-\rho)^2}. \tag{7}$$

The left-hand expression is simply $\Sigma_{t=1}^{\infty} tP(t)$, so Eq. (5) becomes

$$\frac{z}{n} = b(1-\rho)^2. \tag{8}$$

We now wish to set t_0 so that the number of nodes living on trees of size $\geq t_0$ is just barely enough to cover the "excess" of the giant component, $\delta=g-1/2$. In that case, t_0 satisfies the property

$$\delta > \frac{z}{n} \sum_{t=t_0+1}^{\infty} tP(t) \tag{9}$$

$$= \frac{z}{n} \sum_{t=t_0+1}^{\infty} \frac{(\rho t)^t e^{-\rho t}}{(t-1)!} \frac{1-\rho}{\rho} \tag{10}$$

$$= \rho e^{-\rho} \frac{z}{n} \sum_{t=t_0}^{\infty} \frac{(t+1)^{t+1}}{t^t} \frac{(\rho t)^t e^{-\rho t}}{t!} \frac{1-\rho}{\rho} \tag{11}$$

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$$> \rho e^{-\rho} \frac{z}{n} \sum_{t=t_0}^{\infty} 2(t_0+1)P(t). \tag{12}$$

Given t_0 satisfying both this and the corresponding lower bound on δ, bisecting the graph is simply a matter of cutting each tree of size $\geq t_0$. Any resulting imbalance in the partitions (from "overstripping") can easily be corrected by transferring enough monomers: there is an extensive number of these, whereas the largest trees are only of size $O(\log n)$. Thus,

$$\frac{w}{n} \leq \frac{z}{n} \sum_{t=t_0}^{\infty} P(t) \tag{13}$$

$$< \frac{\delta(\rho e^{-\rho})^{-1}}{2(t_0+1)} \tag{14}$$

from Eq. (12). In order for the tree-stripping procedure to work, the mantle must contain enough vertices to strip, and so we know that $b \geq \delta$. One can solve Eqs. (1) and (2) numerically to find that this imposes the condition $g < c_0$ with constant $c_0 \approx 0.813$, corresponding to $\alpha \lesssim 2.06$. For $\alpha_c \leq \alpha < 2.06$, it is straightforward to confirm that $\delta(\rho e^{-\rho})^{-1} < \alpha - \alpha_c$, giving

$$\frac{w}{n} < \frac{\alpha - \alpha_c}{2(t_0+1)}. \tag{15}$$

Finally, we bound t_0 itself. A version of Stirling's formula gives the inequality $t! < t^t e^{-t} \sqrt{2\pi t}/(1-1/2t)$, which we apply in Eq. (10),

$$\delta > \frac{z}{n} \sum_{t=t_0+1}^{\infty} \frac{t(\rho t)^t e^{-\rho t}}{t!} \frac{1-\rho}{\rho} \tag{16}$$

$$> \frac{b(1-\rho)^3}{\sqrt{2\pi\rho}} \sum_{t=t_0+1}^{\infty} \frac{\sqrt{t}(1-1/2t)\rho^t e^{-\rho t}}{e^{-t}} \tag{17}$$

$$> \frac{b(1-\rho)^3}{\sqrt{2\pi\rho}} \sum_{t=t_0+1}^{\infty} (\rho e^{1-\rho})^t \tag{18}$$

$$= \frac{b(1-\rho)^3}{\sqrt{2\pi\rho}} \frac{(\rho e^{1-\rho})^{t_0+1}}{1-\rho e^{1-\rho}}. \tag{19}$$

We have just seen that $\alpha - \alpha_c > \delta(\rho e^{-\rho})^{-1}$ in the regime of interest, so

$$\alpha - \alpha_c > \frac{b(1-\rho)^3}{\sqrt{2\pi}\rho^2 e^{-\rho}(1-\rho e^{1-\rho})} (\rho e^{1-\rho})^{t_0+1}. \tag{20}$$

Then letting $\epsilon = \alpha - \alpha_c$,

$$t_0 + 1 > \frac{\log(1/\epsilon) + \log \dfrac{b(1-\rho)^3}{\sqrt{2\pi}\rho^2 e^{-\rho}(1-\rho e^{1-\rho})}}{\log(1/\rho e^{1-\rho})}. \tag{21}$$

It can be verified that the second term in the numerator is monotone increasing in g and is thus bounded below by its value at $g = 1/2$ (the smallest possible giant component size for $\alpha \geq \alpha_c$), which corresponds to $\rho = \log 2$. This gives

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$$t_0 + 1 > \frac{\log(1/\epsilon) - c_1}{\log(1/\rho e^{1-\rho})}, \tag{22}$$

where

$$c_1 = -\log \frac{(1-\log 2)^3}{\sqrt{2\pi}\log 2(1-e\log 2/2)} \approx 1.25. \tag{23}$$

Finally, the denominator in Eq. (22) is also monotone increasing in g, and thus bounded above by its value where the mantle is exhausted at $g=c_0\approx 0.813$, which corresponds to $\rho=1-1/2c_0$. Therefore,

$$t_0 + 1 > \frac{\log(1/\epsilon) - c_1}{c_2}, \tag{24}$$

where

$$c_2 = -\frac{1}{2c_0} - \log\left(1 - \frac{1}{2c_0}\right) \approx 0.339. \tag{25}$$

Inserting this t_0 bound into Eq. (15) gives the bisection width bound,

$$\frac{w}{n} < \frac{c_2}{2}\frac{\epsilon}{\log(1/\epsilon) - c_1}. \tag{26}$$

We stress several points regarding this bound. First of all, the result closes the gap between upper and lower bounds on w/n at $\alpha=\alpha_c$. The unsurprising fact that both bounds go to zero and the bisection width is continuous at the transition is already apparent in Eq. (14). The less obvious property that its first derivative (with respect to α) is also continuous at the transition follows from Eq. (26). Second of all, the denominator of Eq. (26) is only positive for $0<\epsilon<1/e^{c_1}$. Numerically, this is roughly $1.39<\alpha<1.67$, which covers a significant fraction of the regime of interest $1.39<\alpha<2.06$. Finally, from Eq. (24) it is clear that t_0 diverges at the transition, and for sufficiently small ϵ, it satisfies the even simpler bound $t_0>\log(1/\epsilon)$. These facts will be significant for the arguments in Sec. III.

III. SOLUTION CLUSTER STRUCTURE

We now consider the geometry of the solution space for random graph bisection. For a graph $G\in\mathcal{G}_{np}$, given two distinct optimal bisections A and B, define the relative Hamming distance $\mu(A,B)$ to be the fraction of nodes that are in one of the subsets (V_1 or V_2) in solution A but in the opposite subset in solution B. Since there is a global symmetry under exchange of V_1 and V_2, we are interested in $d(A,B)=\min(\mu(A,B),1-\mu(A,B))$.

Define A and B to be r-adjacent if $d(A,B)\leq r$. For some small r, any two solutions that are connected by a chain of r-adjacent solutions are said to be in the same *solution cluster*. Asymptotically, for a given mean degree α, we would like to know what the limiting cluster structure is when r becomes an arbitrarily small constant. One scenario that is common for sufficiently underconstrained combinatorial optimization problems is *replica symmetry*: for any finite $r>0$, all solutions are with high probability within a single cluster. Another scenario, common for more constrained instances, is *one-step RSB*: for sufficiently small r, there is a large number of disconnected clusters, separated from each other by at least some finite relative Hamming distance.

In what follows, if solutions constructed in a particular way are with high probability r-adjacent for any finite $r>0$, we will refer to them as being simply *adjacent*.

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A. $\alpha < 2 \log 2$

Consider graphs where optimal bisections require no cuts in the giant component. We first argue that we can almost surely create such a bisection by populating one of the two subsets only with the largest component and with monomers and dimers (components of size of 2). We then use arguments in Ref. 19 to show that having created a bisection in this way, it must be in the same cluster as *all* optimal bisections.

As discussed in Sec. II, in $\mathcal{G}_{np}$ graphs for large n, a fraction $e^{-\alpha}$ of nodes are monomers. What about dimers? For a pair of nodes to be a dimer, it must contain an edge (probability $p \approx \alpha/n$), and all other pairs involving one of its two endpoints ($2n-4$ of these) must not contain an edge (probability $1-p$): asymptotically, this occurs with probability $\alpha e^{-2\alpha}/n$. So the expected number of dimer pairs is $\approx \alpha e^{-2\alpha} n/2$, and a.a.s. the fraction of nodes in dimers is $\alpha e^{-2\alpha}$. Thus, a.a.s. the fraction of nodes in monomers or dimers is $s = e^{-\alpha}(1+\alpha e^{-\alpha})$.

For $\alpha \leq 1$, $s > 1/2$. For $1 < \alpha < 2 \log 2$, s falls below 1/2, but one may verify that $g+s > 1/2$, where g is the size of the giant component given in Eq. (1). Therefore, as long as $\alpha < 2 \log 2$, we a.a.s. surely have enough monomers and dimers to fill up a subset that otherwise contains only the largest component (or one of them, if not unique). Call such optimal solutions *clean bisections*.

Next, we show that any two distinct clean bisections A and B (and hence all clean bisections) must be in the same solution cluster. If the largest component is unique, A and B can differ at most by the choice of which monomers and dimers are selected to be in its subset. If the largest component is not unique, it cannot be a giant component, so two clean bisections can then differ at most by (1) a component consisting of a vanishing fraction of nodes and (2) monomers and dimers. In both cases, we can take any one of the differing components in A and swap it with monomers and dimers from the other subset, producing an adjacent clean bisection. By iterating the process, we will necessarily arrive at B through a chain of adjacent clean bisections, since we have already shown that there is a sufficient supply of monomers and dimers for this purpose.

Finally, it is straightforward to see that any optimal solution must be in the same cluster as some clean bisection. This holds for almost the same reason as in the previous paragraph. If the subset containing the largest component also contains any components other than monomers and dimers, each of these asymptotically consists of a vanishing fraction of nodes. They may be iteratively swapped with monomers and dimers to produce a chain of adjacent solutions, leading to a clean bisection.

Since all clean bisections are in the same cluster, and any solution is in the same cluster as some clean bisection, all solutions must lie within a single cluster for $\alpha < 2 \log 2$. This is a replica symmetric phase.

B. $\alpha \geq 2 \log 2$

When the bisection width is nonzero, the situation is more complicated. Now we must consider the different ways in which optimal bisections can cut the giant component. We give arguments suggesting that up to some $\alpha_d > \alpha_c$, optimal bisections avoid a central core of the giant component and are restricted to a well-defined periphery. For those graphs, we then argue that all optimal bisections are in a single cluster.

1. The expander core

Call the giant component S and imagine that for any $\alpha \geq 2 \log 2$, some identifiable part of it, $X \subseteq S$, is an *expander*. This means that cutting off any piece X_0 from X requires (asymptotically) at least some finite *expansion*,

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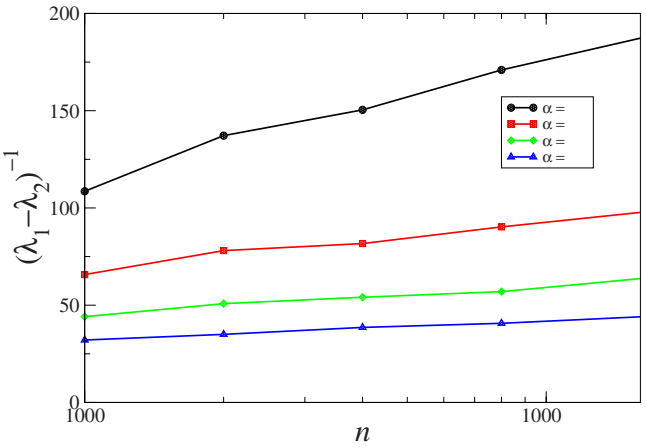


FIG. 3. (Color online) Reciprocal of spectral gap. For mean degree $\alpha=1.4$ through $\alpha=2.0$, on graphs of size $n=1000$ up to $n=16\,000$, the spectral gap vanishes roughly as $\sim 1/\log n$.

$$h = \frac{|E(X \setminus X_0, X_0)|}{\min(|X \setminus X_0|, |X_0|)}, \tag{27}$$

i.e., the number of cut edges per vertex in the broken piece. The expander X would then be closely related to the central core that optimal bisections avoid for $\epsilon=\alpha-\alpha_c$ less than some constant. The reason is as follows. We have seen in Sec. II that when cutting trees *only*, the smallest tree needed is of size

$$t_0 > \log(1/\epsilon), \tag{28}$$

and so the largest expansion is

$$h < \frac{1}{\log(1/\epsilon)}. \tag{29}$$

For sufficiently small ϵ, this must fall below the (finite) value of h for the expander. A consequence is that trimming the giant component's excess requires fewer cuts if these are all performed within the mantle than if even a small fraction of the core is cut.

One might hope that the 2-core is an expander, in which case greedily trimming trees could, in fact, be the optimal way to bisect a graph. As it happens, this is almost certainly not true. Consider the 2-core's *spectral gap,* the difference $\lambda_1-\lambda_2$ between the two largest eigenvalues of its connectivity matrix. Via Cheeger's inequality,[10] this yields an upper bound on expansion. Figure 3 suggests that the spectral gap likely vanishes (albeit slowly) as $n\to\infty$, implying that the 2-core's expansion vanishes, presumably due to the existence of cycles of length $\sim\log n$. Certain "bottlenecks" in the 2-core could then provide more efficient cuts than trees in the mantle.

But some further intuition comes from considering *random* cuts in the 2-core. Figure 4 shows the minimum expansion from a number of ways of randomly slicing the 2-core into two pieces. These numerical results, which we originally reported in Ref. 17, suggest that for large n the minimum expansion never falls below a constant. This means that any bottlenecks that do exist in the 2-core are not immediately apparent, and some significant part of the 2-core could, in fact, be an expander.

Subsequent results by Benjamini *et al.*[2] have confirmed that this is indeed the case: for all $\alpha>1$, they give an explicit procedure for stripping the giant component down to an expander core. The giant component is a "decorated expander," with many small [$O(\log n)$ at the largest] "deco-

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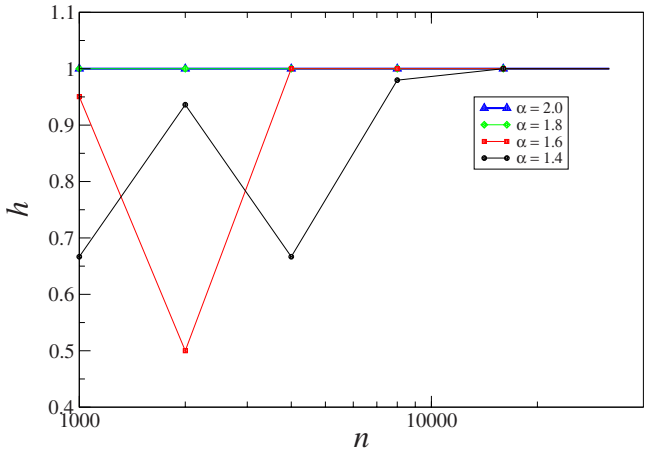


FIG. 4. (Color online) Smallest expansion over a random sample of ways of slicing the 2-core into two pieces. Randomly chosen nodes are sequentially separated from the 2-core until half of the 2-core has been cut. For each value of mean degree α, and for each size from $n=1000$ to $n=32\,000$, this random sequence is repeated ten times on each of 100 graphs. Results show the *lowest* expansion value found over the course of all ten sequences and over all 100 graphs. For increasing graph sizes, this value approaches 1.

rations" D_i attached to the expander core X, as shown schematically in Fig. 5. Therefore, by our previous argument, below some finite ϵ_0, the cuts in all optimal bisections are restricted to the decorations D_i.

2. Consequences for solution structure

We now use arguments related to the ones in Sec. III A to suggest that for $\epsilon<\epsilon_0$ ($\alpha<\alpha_d=2\log 2+\epsilon_0$), all optimal solutions are contained within a single cluster. We first show how to construct a *neat* bisection. By neat, we mean that one of the two subsets contains only the expander core, some attached decorations, and monomers. We then present arguments suggesting that this bisection, even if not optimal, is adjacent to a neat optimal bisection that is in the same cluster as all optimal bisections, thus replica symmetry holds for all $\alpha<\alpha_d$.

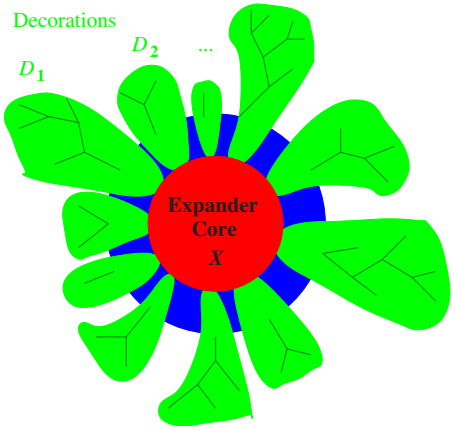


FIG. 5. (Color online) A different view of the giant component in Fig. 2, shown here as an identifiable *expander core X* within the 2-core. The trees are now contained within numerous small *decorations* D_i that are attached to the expander core.

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The construction procedure is as follows. Consider the giant component S as being composed of the expander core X and decorations D_i as in Fig. 5. These are disjoint subsets, with some edges connecting the D_i to X but none connecting the D_i to each other.[2] For each decoration D_i, find the nonempty subset $D_i^1 \subseteq D_i$ that minimizes the expansion

$$h_i^1 = \frac{|E(S \setminus D_i^1, D_i^1)|}{|D_i^1|} \tag{30}$$

resulting from cutting off the piece D_i^1. The value of h_i^1 may be interpreted as a "cost" per node that we attribute to each node in D_i^1. Now repeat this procedure on $D_i \setminus D_i^1$, resulting in a new piece D_i^2, except that this time we find the D_i^2 minimizing the *marginal* cost of cutting off D_i^2 given that D_i^1 is already cut off, i.e.,

$$h_i^2 = \frac{|E(S \setminus (D_i^1 \cup D_i^2), D_i^1 \cup D_i^2)| - |E(S \setminus D_i^1, D_i^1)|}{|D_i^2|}. \tag{31}$$

Recursively perform the procedure, letting

$$h_i^j = \frac{|E(S \setminus \cup_{l=1}^{j} D_i^l, \cup_{l=1}^{j} D_i^l)| - h_i^{j-1}|D_i^{j-1}|}{|D_i^j|}, \tag{32}$$

until decoration D_i has been completely partitioned into disjoint subsets $D_i^1, \ldots, D_i^m$. It is straightforward to confirm that these, by construction, are now ordered in increasing marginal cost $h_i^1 \leq \cdots \leq h_i^m$, where the total cost $|E(S, D_i)|$ of cutting the entire decoration is simply equal to the sum of the marginal costs over all nodes,

$$|E(S, D_i)| = \sum_{j=1}^{m} h_i^j |D_i^j|. \tag{33}$$

Note that h_i^j is the *minimum possible* cost of cutting a node in D_i^j. Since cuts in optimal solutions are restricted to decorations, generating a neat optimal bisection requires an aggregated ranking of all subsets of all decorations D_i^j according to their costs h_i^j. Analogously to the greedy algorithm in Sec. II, where we stripped trees in increasing order of expansion, we now strip the pieces of the decorations until the giant component's excess has been removed. As with the greedy algorithm, this can result in overstripping. There may also be numerous pieces with an equal ranking, leaving the choice of which ones to pick in order to overstrip as little as possible. But due to the maximal size of the decorations,[2] the largest of these pieces is $O(\log n)$. So regardless of which we choose, as with tree stripping, we can compensate for the imbalance by transferring some of the extensive supply of monomers. The result is a neat bisection, albeit not necessarily an optimal one.

In principle, by perfectly selecting the final pieces of equal rank and thereby minimizing the cost of overstripping, we can even form any neat *optimal* bisection in this way. We discuss this in Sec. IV. But no matter how we select the final pieces, it seems highly likely that our construction gives a solution differing from some optimal one by at most the number of variables contained in a finite number of these pieces, which is $O(\log n)$. Furthermore, it is not difficult to see that all neat (not necessarily optimal) bisections that use the identical choice of compensating monomers lie within a single cluster. It then follows that among these neat bisections, all those that are optimal are in a single cluster as well. Additionally, *every* neat optimal bisection must then be in the same cluster, since successively swapping monomers give a chain of adjacent solutions connecting them.

Finally, for similar reasons as in Sec. II, any optimal bisection must be in the same cluster as some neat optimal bisection. If the subset containing the expander core also contains components other than monomers, each one can have size at most $O(\log n)$ and can be swapped with monomers from the other subset to produce an adjacent optimal bisection. Iterating this process leads to

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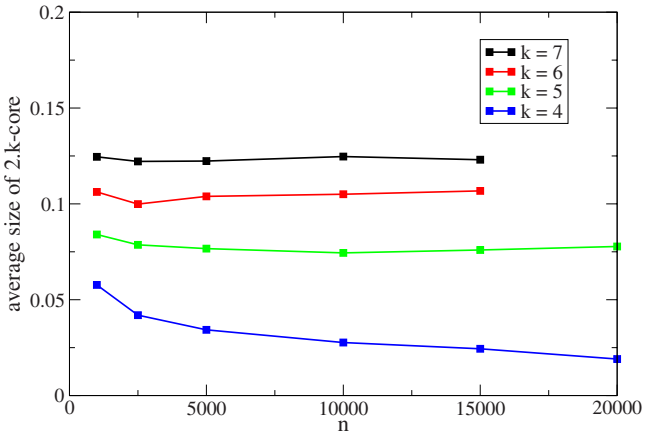


FIG. 6. (Color online) Fraction of nodes in the 2.k-core, $k=4$ through $k=7$, at $\alpha=1.4$. Numerical results are averaged over 200 instances, for increasing values of n.

a neat optimal bisection. Consequently, the argument implies that for $2\log 2<\alpha<\alpha_d$, all optimal bisections lie within a single cluster, and the replica symmetric phase extends up to $\alpha_d>\alpha_c$.

Above α_d, the solution structure is less clear. When α is large enough, optimal cuts are no longer necessarily restricted to the decorations: the expander core likely needs to be cut as well. In that case, the sizes of the cut pieces could well be extensive. One could easily imagine extensive gaps in the solution structure, leading to multiple clusters and RSB. Although we have no direct evidence of it, we speculate that this is the case for α greater than some finite value, suggesting a RSB phase boundary at or above α_d, i.e., above the critical threshold α_c.

IV. DISCUSSION AND CONCLUSIONS

Our study points to an unusual kind of phase structure for random graph bisection. The typical scenario, in random combinatorial optimization problems displaying a sharp threshold in the objective function's behavior, is that a RSB transition takes place at or below the main critical threshold: $\alpha_d\leq\alpha_c$. By contrast, graph bisection appears to be replica symmetric up to *and beyond* the threshold, with $\alpha_d>\alpha_c$. We note that this lends support to an analytical bisection width prediction proposed by Liao in 1987.[24,25] Let us now comment on several further aspects of α_d, as well as on the intriguing algorithmic consequences of this phase picture.

Although it appears that α_d is strictly greater than α_c, it may not be much greater. As mentioned at the end of Sec. II, our bisection width bound is only valid for $\alpha<1.67$. That bound is needed for being able to restrict optimal cuts to the giant component's decorations. We do not presently have any other quantitative bound or estimate on the specific value of α_d. However, some further qualitative observations come from examining the procedure for obtaining the expander core. Roughly speaking, Benjamini *et al.*[2] stripped the giant component down to what we call a 2.k-core, namely, a connected subgraph of the giant component where all vertices have degree at least 2 (no trees) and where the largest chain is of length k. The 2.k-core is an expander as long as k is a sufficiently large finite constant. The construction in Ref. 2 does not explicitly state how large k must be or what the minimum expansion value h is for the 2.k-core at a given mean degree α. But decreasing values of k clearly result in smaller 2.k-cores (Fig. 6) and presumably in larger expansion as more long chains are removed. This would lead to a higher α_d. Thus, by picking the smallest possible k (for every α) that still yields an expander core, the resulting expansion gives, via Eq. (29), the largest possible α_d for which we can be sure that graph bisection is replica symmetric at all $\alpha<\alpha_d$.

Algorithmically, the consequence of our study is that we may be able to obtain near-optimal bisections and perhaps even optimal ones, for all $\alpha<\alpha_d$, in polynomial time. By near-optimal, we mean that the ratio of its bisection width to the optimal value approaches 1 asymptotically. We do

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this by stripping the giant component down to the expander core (using any sufficiently large k), and then following the construction in Sec. III B 2. The stripping process and decoration identification[2] take polynomial time. The next subtlety is breaking a decoration D_i into the correct disjoint pieces D_i^j: an exhaustive approach to finding D_i^1 involves comparing all of the $2^{|D_i|}$ ways of partitioning D_i into two subsets. But since no decoration D_i has more than $O(\log n)$ nodes, this can be done in $n^{O(1)}$ steps, and then all pieces D_i^j can be found and ranked $(h_i^1 \leq \cdots \leq h_i^m)$ in $n^{O(1)}$ steps. For the final pieces of equal rank, even if we simply pick them in arbitrary order we will overstrip by at most $O(\log n)$. Thus, in polynomial time we conceivably reach a solution whose (extensive) bisection width is within $O(\log n)$ of the optimal one.

Of course, the $O(1)$ constant in the exponent may be quite large, which could make this result largely a theoretical one. But it is worth noting that we have some freedom to decrease the size of the decorations, and thus lower the constant. If, say, instead of using the smallest possible k that gives an expander core, we use the *largest* possible k that gives expansion greater than our tree-cutting bound [Eq. (29)], this will strip the minimum possible from the giant component. In this way, we may be able to minimize the size of the largest decoration.

It would be particularly remarkable if some version of our construction could give a true optimal bisection, with high probability, in polynomial time. For this, we need to pick the final pieces of equal rank so as to minimize overstripping. The difficulty is that the number of such pieces is likely to be infinite: an extensive number of nodes is contained collectively within the decorations, while there are at most $O(\log^2 n)$ possible marginal costs [$O(\log n)$ possible integers for the numerator and $O(\log n)$ possible integers for the denominator]. On the other hand, it could well be that of the infinite number of final pieces, a finite fraction of them have any given finite size $|D_i^j|$. In that case, a simple greedy procedure will find a collection of pieces that minimizes overstripping. Having minimized overstripping, one still needs to check whether it is possible to lower the bisection width further by restoring a small fragment of a cut piece. If the amount we overstrip is $O(\log n)$, then exhaustively checking this could require $\sim \binom{n}{\log n}$ steps. But more likely, one can strengthen our analysis to show that the final pieces are, in fact, of size $O(1)$, in which case the last operation could also be accomplished in polynomial time. Clearly, this intuition is far from a proof of any sort. But if indeed the optimal bisection can be found in polynomial time for $\alpha < \alpha_d$, we would have a very unusual example of an NP-hard optimization problem whose typical-case complexity is polynomial at and *above* the main threshold α_c.

We offer two final remarks about other versions of the problem. First, our approach suggests that the peculiarities of random graph bisection's phase structure are closely related to the problem's global constraint—the fact that a subset cannot be larger than $n/2$—combined with the specific topology of the giant component in $\mathcal{G}_{np}$. This means that the problem of balanced multiway graph partitioning on $\mathcal{G}_{np}$ graphs should have very similar characteristics and can likely be analyzed through similar means. Second, the situation is very different if instead of $\mathcal{G}_{np}$ we use a graph ensemble that has local geometric structure. An example is the random geometric graph ensemble $\mathcal{G}_{nr}$,[31] where nodes are placed uniformly at random in a unit square and pairs are connected by an edge if separated by distance $\leq r$. In this case, the giant component is very unlikely to have any expander core due to its large diameter. Rather than having a treelike periphery, it contains numerous local cliques, and minimizing the bisection width involves locating the bottlenecks in between those cliques. But at the same time, $\mathcal{G}_{nr}$ graphs have useful lattice-like properties,[15] with the optimal bisection width scaling[31,7] as $w \sim \sqrt{n}$, precisely as one finds on a regular two-dimensional lattice structure. We expect that the structure of bisections is closely related to the geometry of the giant component in bond percolation on a lattice, which has been studied.[3] Adapting that analysis could lead to nontrivial bounds on w and rapid algorithms finding very close approximations to the optimal cut in $\mathcal{G}_{nr}$.

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